ON SOME MODULI SPACES OF STABLE VECTOR BUNDLES ON CUBIC AND QUARTIC THREEFOLDS

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ABSTRACT. We study certain moduli spaces of stable vector bundles of rank two on cubic and quartic threefolds. In many cases under consideration, it turns out that the moduli space is complete and irreducible and a general member has vanishing intermediate cohomology. In one case, all except one component of the moduli space has such vector bundles.

1. INTRODUCTION

A vector bundle E on a smooth hypersurface $X \subset \mathbb{P}^n$ is called *arithmetically Cohen-Macaulay* (*ACM* for short) if

$$\mathrm{H}^i_*(X, E) := \bigoplus_{\nu \in \mathbb{Z}} \mathrm{H}^i(X, E(\nu)) = 0$$

for all $0 < i < n-1$, where $E(\nu) := E \otimes_{\mathcal{O}_X} \mathcal{O}_X(\nu)$. The study of ACM vector bundles of rank two on hypersurfaces has received significant attention in the recent past (see [1, 6, 7, 8, 17, 20, 21, 22, 25, 3]).

We note an interesting property of such bundles: Consider the finitely generated graded $\mathrm{H}^0_*(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$−module $\mathrm{H}^0_*(X, E)$. Any choice of a minimal set of generators $g_i \in \mathrm{H}^0(X, E(a_i))$, $1 \le i \le \ell$, gives a minimal resolution

$$0 \to F_1 \xrightarrow{\Phi} F_0 \to E \to 0,$$

where $F_0 := \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}^n}(-a_i)$ and the map $F_0 \to E$, which yields a surjection $\mathrm{H}^0_*(F_0) \twoheadrightarrow \mathrm{H}^0_*(X, E)$, is defined by the generators $\{g_i\}$. It follows from the Auslander-Buchsbaum formula (see [12], Chapter 19) that F_1 is a vector bundle on $\mathbb{P}^n$. It is easily verified that F_1 is ACM and hence by Horrocks' criterion (see [24], page 39) splits into a direct sum of line bundles. By changing bases of F_1 and F_0, the homomorphism Φ can be chosen to be skew-symmetric, and it can be checked that the Pfaffian of Φ is the polynomial defining X (see [1] for details).

We will briefly describe what is known about ACM vector bundles of rank two on hypersurfaces.

• Any ACM vector bundle of rank two on a general hypersurface $X \subset \mathbb{P}^4$ (respectively, $X \subset \mathbb{P}^{n \ge 5}$) of degree $d \ge 6$ (respectively, $d \ge 3$), is a direct sum of line bundles (see [20, 21, 25]). Equivalently, any codimension two *arithmetically Gorenstein* subscheme (i.e. the zero locus of a section of an ACM bundle of rank two) in a general smooth hypersurface $X \subset \mathbb{P}^n$ for $n \ge 5$, $d \ge 3$ or $n = 4$, $d \ge 6$ is a complete intersection.

As explained in [25], this can be viewed as a "generalised Noether-Lefschetz theorem" for curves in hypersurfaces in $\mathbb{P}^4$. On the other hand, this also provides a verification of the first non-trivial case of a (strengthening of a) conjecture of Buchweitz, Greuel and Schreyer (see Conjecture B of [4]).

• A general quintic threefold supports only finitely many indecomposable ACM bundles (see [20]). The zero loci of non-zero sections of these ACM bundles define non-trivial cycles in the Griffiths group of the quintic (see [25]).

• Any ACM bundle on a smooth quadric hypersurface is isomorphic to a direct sum of line bundles and spinor bundles (see [16]).

Recall that a vector bundle E over X is called *normalised* if $h^0(E(-1)) = 0$ and $h^0(E) \neq 0$. The first Chern class of a normalised ACM bundle E of rank two satisfies the inequality: $3 - d \leq c_1(E) \leq d - 1$, where $d := \deg X$ (see [17, 25]).

Here we take up the study of indecomposable, normalised ACM vector bundles of rank two on smooth cubic and quartic hypersurfaces in $\mathbb{P}^4$. These hypersurfaces contain a line, which via Serre's construction (see [24], pages 90–94) yields an indecomposable, ACM vector bundle of rank two. The condition of being ACM then implies, by the Grothendieck-Riemann-Roch formula, that the second Chern class c_2 is a function of c_1. The main results here are explicit descriptions of the moduli space of (semi-)stable bundles with the above Chern classes. This is done by showing that at least in many cases, any stable bundle of rank two with the above Chern classes is in fact ACM.

When $c_1(E) = d-1$, the existence of an indecomposable ACM vector bundle E of rank two on a smooth hypersurface X of degree d, is equivalent to X being the Pfaffian of a skew-symmetric matrix of size $2d \times 2d$ with linear entries. Starting from this point of view, the moduli spaces of stable vector bundles with $c_1(E) = d - 1$ on smooth cubic and quartic threefolds have been extensively studied (see [18, 14, 11, 15]). It turns out that in the cubic case, this moduli space is irreducible and contains only ACM bundles while in the quartic case there are components of the moduli spaces which contain only ACM bundles. Using the properties of ACM bundles, one can compute their dimension and prove smoothness of these "ACM" components.

In what follows, we complete the picture by studying ACM bundles for the remaining (possible) values of $c_1(E)$. Let $X \subset \mathbb{P}^4$ be a smooth hypersurface. The inclusion of X in $\mathbb{P}^4$ induces isomorphisms $\mathrm{H}^{2i}(X, \mathbb{Z}) = \mathrm{H}^{2i}(\mathbb{P}^4, \mathbb{Z})$ for $i = 1, 2$. Since $\mathrm{H}^{2i}(\mathbb{P}^4, \mathbb{Z}) = \mathbb{Z}$, where $i = 1, 2$, both $\mathrm{H}^2(X, \mathbb{Z})$ and $\mathrm{H}^4(X, \mathbb{Z})$ will be identified with $\mathbb{Z}$. Let $\mathcal{M}_X(2; c_1, c_2)$ denote the moduli space of normalised, (semi-)stable vector bundles of rank two with $(c_1(E), c_2(E)) = (c_1, c_2)$.

The main results are the following:

Theorem 1. *Let $X \subset \mathbb{P}^4$ be a smooth cubic hypersurface. Let $F(X)$ denote the Fano variety of lines on X. There are canonical isomorphisms*

$$\mathcal{M}_X(2; 1, 2) \cong \mathcal{M}_X(2; 0, 1) \cong F(X).$$

Hence these are smooth irreducible surfaces.

Theorem 2. *Let $X \subset \mathbb{P}^4$ be a smooth quartic hypersurface. Let $F(X)$ denote the Fano variety of lines on X and let $F_2(X)$ denote the Hilbert scheme of plane conics in X.*

(1) *Let X be a general quartic threefold. Let $\mathcal{M}$ denote any moduli component of $\mathcal{M}_X(2; 2, 8)$ which contains only ACM bundles. Then $\mathcal{M}$ is smooth of dimension 5.*
(2) *There is an isomorphism*

$$\mathcal{M}_X(2; 1, 3) \cong F(X).$$

Hence, for X general, this moduli space is a smooth irreducible curve.
(3) *There is an isomorphism*

$$\mathcal{M}_X(2; 0, 2) \cong F_2(X).$$

Hence, for X general, the moduli space $\mathcal{M}_X(2; 0, 2)$ is a smooth irreducible surface.

2. THE CUBIC THREEFOLD

Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold.

Lemma 1. *Any normalised stable vector bundle $E \to X$ of rank two with*

$$(c_1(E), c_2(E)) = (c_1, c_2) = (1, 2)$$

is indecomposable and ACM.

Proof. Any hypersurface on X is linearly equivalent to $\mathcal{O}_X(j)$ for some $j > 0$. Since E is normalised, we have $\mathrm{H}^0(X, E(-1)) = 0$. Hence the zero locus of any non-zero section of the rank two vector bundle E is either empty or pure of codimension 2. If E has a nowhere vanishing section, then this implies that E is a direct sum of line bundles and this contradicts stability.

Let $s \in \mathrm{H}^0(X, E)$ be a non-zero section whose zero locus C is pure of codimension 2 in X. So $\deg C = c_2(E) = 2$. Let $S \subset X$ be a general hyperplane section so that $\Gamma := C \cap S$ is a zero-dimensional subscheme of length two. Notice that Γ is the zero locus of the image of s under the map $\mathrm{H}^0(E) \hookrightarrow \mathrm{H}^0(E_{|S})$.

We will show that E is ACM if the restriction $E_{|S}$ is ACM. To prove this, we proceed as follows: Consider the short exact sequence

$$0 \to E(n-1) \to E(n) \to E(n)_{|S} \to 0$$

on X. The homomorphism $\mathrm{H}^1(X, E(n-1)) \longrightarrow \mathrm{H}^1(X, E(n))$ in the corresponding long exact sequence of cohomology groups is a surjection. Since $\mathrm{H}^1(X, E(n)) = 0$ for $n << 0$, this implies that $\mathrm{H}^1(X, E(n)) = 0$ for all $n \in \mathbb{Z}$. By Serre duality we have $\mathrm{H}^2(X, E(n)) = \mathrm{H}^1(X, E(-3-n))^* = 0$. Hence E is ACM if $E_{|S}$ is ACM.

To prove that $E_{|S}$ is ACM, we first claim that $h^2(E_{|S}(-1)) = 0$. By Serre duality,

$$h^2(E_{|S}(-1)) = h^0(E_{|S}(-1)).$$

Since $h^0(S, \mathcal{O}_S(-1)) = 0$, in view of the short exact sequence

$$(1) \qquad\qquad 0 \longrightarrow \mathcal{O}_S \xrightarrow{\cdot s|_S} E_{|S} \longrightarrow I_{\Gamma/S}(1) \longrightarrow 0$$

given by $s|_S$, it is enough to show that $h^0(I_{\Gamma/S}) = 0$. Consider the exact sequence

$$0 \to I_{\Gamma/S} \to \mathcal{O}_S \to \mathcal{O}_\Gamma \to 0.$$

The corresponding induced map $\mathbb{C} \cong \mathrm{H}^0(\mathcal{O}_S) \to \mathrm{H}^0(\mathcal{O}_\Gamma) \cong \mathbb{C}^2$ is injective. Consequently, $h^2(E_{|S}(-1)) = 0$ and hence $h^2(E_{|S}(k)) = 0$ for $k \geq 0$.

Now the long exact sequence of cohomologies associated to the short exact sequence

$$0 \to I_{\Gamma/S}(1) \to \mathcal{O}_S(1) \to \mathcal{O}_\Gamma(1) \to 0$$

yields:

$$0 \to \mathrm{H}^0(I_{\Gamma/S}(1)) \to \mathrm{H}^0(\mathcal{O}_S(1)) \to \mathrm{H}^0(\mathcal{O}_\Gamma(1)) \to \cdots.$$

The line bundle $\mathcal{O}_S(1)$ being very ample, separates points and tangents of S. Since the length of the zero-dimensional scheme Γ is two, the restriction homomorphism

$$\mathrm{H}^0(S, \mathcal{O}_S(1)) \to \mathrm{H}^0(\Gamma, \mathcal{O}_\Gamma(1))$$

is surjective. Thus $h^0(I_{\Gamma/S}(1)) = 2$, $h^0(E_{|S}) = 3$ and $h^1(I_{\Gamma/S}(1)) = 0$. Since $h^1(\mathcal{O}_S) = 0$, it follows from the short exact sequence (1) that $h^1(E_{|S}) = 0$. Combining this with the fact that $h^2(E_{|S}(-1)) = 0$, we see that $E_{|S}$ is $1-$regular in the sense of Castelnuovo and Mumford. Hence

$$h^1(E_{|S}(n)) = 0$$

for $n \geq 0$. By Serre duality,

$$h^1(E_{|S}(-n-2)) = 0 \quad \forall\, n \geq 0.$$

It remains to be shown that $h^1(E_{|S}(-1)) = 0$. The Riemann-Roch Theorem for $E_{|S}$ says

$$\chi(E_{|S}) = \frac{3}{2}c_1^2 + \frac{3}{2}c_1 - c_2 + 2.$$

Since $h^0(E_{|S}(-1)) = 0 = h^2(E_{|S}(-1))$ and $c_1(E_{|S}(-1)) = -1$, $c_2(E_{|S}(-1)) = 2$, by Riemann-Roch we have $0 = \chi(E_{|S}(-1)) = h^1(E_{|S}(-1))$. Therefore, $E_{|S}$ is ACM, completing the proof of the lemma. □

The following lemma gives a minimal resolution of any stable vector bundle E in the moduli space $\mathcal{M}_X(2; 1, 2)$.

Lemma 2. *A vector bundle E in $\mathcal{M}_X(2; 1, 2)$ has the following minimal resolution*

$$(2) \qquad 0 \to \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^4}(-1) \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^4}(-1) \to E \to 0.$$

Proof. Take any non-zero section s of E. We have the short exact sequence

$$(3) \qquad\qquad 0 \longrightarrow \mathcal{O}_X \overset{\cdot s}{\longrightarrow} E \longrightarrow I_{C/X}(1) \longrightarrow 0,$$

where $C \subset X$ as before is the zero locus of s. Since $h^0(E) = h^0(E_{|S}) = 3$, we have $h^0(I_{C/X}(1)) = 2$ and so $C \subset \mathbb{P}^3$. Now $I_{C/X}$ is ACM $\implies I_{C/\mathbb{P}^4}$ is ACM $\implies I_{C/\mathbb{P}^3}$ is ACM. Thus $C \subset \mathbb{P}^4$ is a complete intersection of type $(1, 1, 2)$ and so there is a surjection

$$\mathcal{O}_X(-1)^{\oplus 2} \oplus \mathcal{O}_X(-2) \twoheadrightarrow I_{C/X}.$$

It is straightforward to see this surjection lifts to a map

$$\mathcal{O}_X^{\oplus 2} \oplus \mathcal{O}_X(-1) \to E.$$

Recall that E has a resolution

$$0 \to F_1 \to F_0 \to E \to 0,$$

where F_0, F_1 are direct sums of line bundles, and $F_1 \cong F_0^\vee(-2)$. Since $h^0(F_0) = h^0(E) = 3$, $h^0(F_1) = 0$ and so we have $F_0 = \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus l} \oplus \mathcal{O}_{\mathbb{P}^4}^3$ where $l \in \{1, 3\}$. It is not hard to see that the case $l = 3$ is not possible. □

Lemma 3. $\mathrm{H}^2(X, \mathcal{E}nd E) = 0.$

Proof. We have an exact sequence

$$0 \to \mathcal{O}_X \to E \to I_{C/X}(1) \to 0.$$

Tensoring this with $E^\vee$, we get $\mathrm{H}^2(X, \mathcal{E}nd E) \cong \mathrm{H}^2(E \otimes I_{C/X})$. From the exact sequence

$$0 \to E \otimes I_{C/X} \to E \to E_{|C} \to 0$$

we see that $\mathrm{H}^2(E \otimes I_{C/X}) \cong \mathrm{H}^1(E_{|C}) = \mathrm{H}^1(N_{C/X})$. By Serre duality, this is isomorphic to $\mathrm{H}^0(E(-2)_{|C})$. To compute the latter, we apply the functor $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^4}}(-, \mathcal{O}_C(-1))$ to the minimal resolution (2) to get a sequence

$$0 \to \mathrm{H}^0(E^\vee(-1)_{|C}) \to \mathrm{H}^0(\mathcal{O}_C(-1))^{\oplus 3} \oplus \mathrm{H}^0(\mathcal{O}_C) \to \mathrm{H}^0(\mathcal{O}_C(1))^{\oplus 3} \oplus \mathrm{H}^0(\mathcal{O}_C).$$

The map $\mathrm{H}^0(\mathcal{O}_C) \to \mathrm{H}^0(\mathcal{O}_C(1))^{\oplus 3}$ is clearly injective: this is because the map is induced by linear entries in the matrix not all of which vanish on C. Thus we have $\mathrm{H}^0(E^\vee(-1)_{|C}) \cong \mathrm{H}^0(E(-2)_{|C}) = 0$. □

Let $\mathcal{Q}$ be the Hilbert scheme of all curves in X which are complete intersections of type $(1, 1, 2)$ in $\mathbb{P}^4$. Consider the Abel-Jacobi map

$$\alpha : \mathcal{Q} \to \mathrm{J}^2(X)$$

which associates with the curve C the class of the cycle $3C - 2h^2$. This map α coincides with the composition

$$\mathcal{Q} \xrightarrow{p} \mathcal{M}_X(2; 1, 2) \xrightarrow{c_2} \mathrm{J}^2(X),$$

where the map p, whose fibre at $[E] \in \mathcal{M}_X(2; 1, 2)$ is $\mathbb{P}(\mathrm{H}^0(X, E)) \cong \mathbb{P}^2$, is given by the Serre construction, while the map c_2 sends any vector bundle E to the image of the Grothendieck-Chern class $3c_2(E) - 2c_1(E)^2$ by the Abel-Jacobi map.

Since $\mathrm{H}^1(C, N_{C/X}) = 0$, the Hilbert scheme $\mathcal{Q}$ is smooth. Furthermore, by Riemann-Roch, we have $h^0(N_{C/X}) = \deg N_{C/X} + 2(1 - g)$. Here $g = 0$ and $\deg N_{C/X} = 2$. Thus $h^0(N_{C/X}) = 4$ and so $\dim(\mathcal{Q}) = 4$ as a result of which we have $\dim(\mathcal{M}_X(2; 1, 2)) = 2$.

Lemma 4. *The moduli spaces $\mathcal{M}_X(2; 1, 2)$ and $\mathcal{Q}$ are smooth and the map*

$$c_2 : \mathcal{M}_X(2; 1, 2) \to \mathrm{J}^2(X)$$

is quasi-finite and étale onto its image.

Proof. By Lemma 3, both $\mathcal{M}_X(2; 1, 2)$ and $\mathcal{Q}$ are smooth. To check that the map c_2 is smooth, we will use results of Welters (see [26]): The tangent map

$$\tau : \mathrm{T}_{[C]}(\mathcal{Q}) \to \mathrm{T}_{[3C - 2h^2]} \mathrm{J}^2(X)$$

for α is a homomorphism of vector spaces $\mathrm{H}^0(C, N_{C/X}) \to \mathrm{H}^1(\Omega_X^2)^\vee$; using Serre duality and the isomorphism $N_{C/X}^\vee \cong N_{C/X}(-1)$, we get its dual map

$$\tau^* : \mathrm{H}^1(\Omega_X^2) \to \mathrm{H}^1(C, N_{C/X}((-2)).$$

This homomorphism fits into a commutative diagram:

$$
\begin{array}{ccc}
\mathrm{H}^0(X, \mathcal{O}_X(1)) & \xrightarrow{R} & \mathrm{H}^1(\Omega_X^2) \\
r_C \downarrow & & \downarrow \tau^* \\
\mathrm{H}^0(C, \mathcal{O}_C(1)) & \xrightarrow{\partial} & \mathrm{H}^1(C, N_{C/X}(-2))
\end{array}
$$

where ∂ is the coboundary map in the long exact sequence of cohomologies associated to the short exact sequence

$$0 \to N_{C/X} \to N_{C/\mathbb{P}^4} \to \mathcal{O}_C(3) \to 0,$$

and R is the coboundary map $\mathrm{H}^0 \to \mathrm{H}^1$ for the exact sequence

$$0 \to \Omega_X^2 \to \Omega_{\mathbb{P}^4}^3 \otimes N_{X/\mathbb{P}^4} \to \Omega_X^3 \otimes N_{X/\mathbb{P}^4} \to 0.$$

It is known (see *op. cit.*) that R is an isomorphism and r_C is a surjection (from a 5-dimensional vector space onto a 3-dimensional vector space). We shall now show that $\dim \mathrm{Image}(\tau^*) = 2$. Since $\mathrm{Image}\, \tau^* = \mathrm{Image}\, \partial$, it is enough to prove that $\dim \mathrm{Image}(\partial) = 2$. Since $C \subset \mathbb{P}^4$ is a complete intersection of type $(1, 1, 2)$, $N_{C/\mathbb{P}^4} = \mathcal{O}_C(1)^{\oplus 2} \oplus \mathcal{O}_C(2)$. From the associated long exact sequence of cohomologies one gets a surjection $\mathrm{H}^1(C, N_{C/X}(-2)) \twoheadrightarrow \mathrm{H}^1(N_{C/\mathbb{P}^4}(-2)) \cong \mathbb{C}^2$. Since $\mathrm{H}^1(N_{C/X}(-2)) \cong \mathrm{H}^0(C, N_{C/X})$ is 4-dimensional as seen before, we get $\mathrm{Image}(\partial)$ is 2-dimensional. Thus the map α has a 2-dimensional image and so we are done. $\qquad\square$

Let $F(X)$ denote the Fano variety of lines in X. It is well known (see [5]) that this variety is smooth and irreducible and that its Albanese is isomorphic to the intermediate Jacobian $J^2(X)$. Druel [11] has proved that the compactified moduli space $\overline{\mathcal{M}_X(2; 2, 5)} \cong \widetilde{J^2(X)}$, where $\widetilde{J^2(X)}$ is the intermediate Jacobian $J^2(X)$ blown up along a translate of $F(X)$. The following two

theorems show that the remaining two moduli spaces $\mathcal{M}_X(2; 1, 2)$ and $\mathcal{M}_X(2; 0, 1)$ which are of interest to us, also admit an explicit description thereby proving that they are smooth, and irreducible.

Theorem 3. *The second Chern class map $c_2 : \mathcal{M}_X(2; 1, 2) \to J^2(X)$ induces an isomorphism*

$$\mathcal{M}_X(2; 1, 2) \cong F(X).$$

Proof. The image of the Abel-Jacobi map $\alpha : \mathcal{Q} \to J^2(X)$ is known to be isomorphic to $F(X)$. But we have shown above that this map factors via $\mathcal{M}_X(2; 1, 2)$. Hence we have the induced map $c_2 : \mathcal{M}_X(2; 1, 2) \to F(X)$ which is étale onto its image. We give an explicit description of this map and from this deduce that this is indeed an isomorphism by showing that it has a unique section.

The restriction of the resolution

$$0 \to F_1 \to F_0 \to E \to 0$$

yields a four term exact sequence (see [20])

$$0 \to E(-3) \to \overline{F}_1 \xrightarrow{\overline{\Phi}} \overline{F}_0 \to E \to 0.$$

The image G of $\overline{\Phi}$ is an ACM vector bundle of rank two. It can be easily verified that $G(1)$ has a unique section whose zero locus is a line ℓ. Since the minimal resolution of any E as above is unique up to isomorphism, this map is well defined. To give a section, we just reverse the process. We first observe that $\mathrm{Ext}^1_{\mathcal{O}_X}(I_{\ell/X}, \mathcal{O}_X) \cong \mathbb{C}$. This follows by mimicking the argument given in the proof of Theorem 5.1.1 on pages 94-97 of [24]. As a result, there is a unique vector bundle of rank two which corresponds to ℓ via Serre's construction and hence the inverse image of c_2 has cardinality one. This completes the proof. □

As a corollary, we obtain

Theorem 4. $\mathcal{M}_X(2; 1, 2) \cong \mathcal{M}_X(2; 0, 1)$.

Proof. One checks that $G(1)$ above has Chern classes $(c_1, c_2) = (0, 1)$. This completes the proof. □

Corollary 1 (see also [13], Section 3)**.** *The Hilbert scheme $\mathcal{Q}$ of plane conics contained in X is smooth and irreducible of dimension 4.*

Proof. The result follows from the fact that $\mathcal{Q}$ is a $\mathbb{P}^2$-bundle over $F(X)$. □

3. The quartic threefold

3.1. **Stable Bundles with** $(c_1, c_2) = (2, 8)$**.** The starting point of our investigation is the following result which can be found in [15]:

Proposition 1. *Let X be a smooth quartic threefold. Then the moduli space of normalised, stable vector bundles of rank two with $(c_1, c_2) = (2, 8)$ has finitely many components: There is a unique component containing bundles which via Serre's construction correspond to a union of two plane sections of X, while the remaining are ACM components i.e., contain bundles which are ACM (i.e. instantons).*

Lemma 5. *Let $\mathcal{M}_I$ denote any component of the moduli space which contains ACM bundles. Any bundle E which is in $\mathcal{M}_I$ admits a minimal resolution*

(4) $$0 \to \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2} \to E \to 0.$$

Proof. We first show that the zero locus of any non-zero section of E is a complete intersection of three quadrics in $\mathbb{P}^4$. From Riemann-Roch, it is easy to see that E has 4 sections, hence E is at least 4-generated. Let $C \subset X$ be the zero locus of any non-zero section of E. From the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\cdot s} E \longrightarrow I_{C/X}(2) \longrightarrow 0$$

we see that $h^0(I_{C/X}(2)) = 3$, and hence $C \subset X$ is contained in three mutually independent quadrics $Q_i \subset \mathbb{P}^4$ for $1 \leq i \leq 3$. Since $\deg C = 8$ and $C \subset \cap_{i=1}^3 Q_i$, it follows that $C = \cap_{i=1}^3 Q_i$. Next, we shall show that E is 6-generated with the other two generators in degree 1. To see this notice that since $F_1 \cong F_0^\vee(-2)$, any resolution must be of the form

(5)
$$0 \to \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus k} \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus k} \to E \to 0,$$

where $k \in \{0, 2, 4\}$. If $k = 0$, then this implies that E is 0-regular, and hence $h^0(E) = 0$ which is a contradiction. When $k = 4$, then it can be easily seen that X is not smooth. $\square$

The following result shows that when X is general, the "ACM" components of rank two stable bundles that we are interested in, are all smooth. Note that it is easy to prove this statement for each case by using the minimal resolutions as in the cubic case (see Lemma 3).

Lemma 6. *Let X be a general quartic threefold. For any ACM rank two bundle E,*

$$\mathrm{H}^2(X, \mathcal{E}ndE) = 0.$$

Proof. This follows from results in [20]. First one shows that $\mathrm{H}_*^2(X, \mathcal{E}ndE)$ is a finite length module over the graded ring $\mathrm{H}_*^0(X, \mathcal{O}_X)$ with the generator living in the one-dimensional vector space $\mathrm{H}^2(X, (\mathcal{E}ndE)(-d))$ (see [20, Corollary 2.3 and sequence (6)]. Next it is shown that for any $g \in \mathrm{H}^0(\mathcal{O}_X(d))$, the multiplication map $\mathrm{H}^2(X, (\mathcal{E}ndE)(-d)) \xrightarrow{g} \mathrm{H}^2(X, \mathcal{E}ndE)$ is zero [20, Corollary 3.8]. $\square$

Let $\mathcal{M}$ denote the union of the ACM components $\mathcal{M}_I$. By results in section 3, [20], there are only finitely many such components. This proves in particular that $\mathcal{M}$ is a finite disjoint union of smooth varieties.

Theorem 5. *Let X be a general quartic threefold. Let $\mathcal{Q}$ denote the Hilbert scheme of all curves in X which are complete intersections of three quadrics in $\mathbb{P}^4$. The moduli spaces $\mathcal{M}$, $\mathcal{Q}$ are smooth of dimension 5 and 8 respectively and the map $c_2 : \mathcal{M} \to \mathrm{J}^2(X)$ is étale and quasi-finite.*

Proof. When X is general, by Lemma 6, $\mathcal{M}$ is smooth. Since $\mathcal{Q}$ is a projective bundle over $\mathcal{M}$, it is also smooth. To check that the map c_2 is smooth, we will again use results of Welters (see [26]): The tangent map $\tau : \mathrm{T}_{[C]}(\mathcal{Q}) \to \mathrm{T}_{[C-2h^2]}\mathrm{J}^2(X)$ is a map of vector spaces $\tau : \mathrm{H}^0(C, N_{C/X}) \to \mathrm{H}^1(\Omega_X^2)^\vee$; using Serre duality and the isomorphism $N_{C/X}^\vee \cong N_{C/X}(-2)$, we get its dual map $\tau^* : \mathrm{H}^1(\Omega_X^2) \to \mathrm{H}^1(C, N_{C/X}(-1))$. This map fits into a commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}^0(X, \mathcal{O}_X(3)) & \xrightarrow{R} & \mathrm{H}^1(\Omega_X^2) \\
r_C \downarrow & & \downarrow \tau^* \\
\mathrm{H}^0(C, \mathcal{O}_C(3)) & \xrightarrow{\partial} & \mathrm{H}^1(C, N_{C/X}(-1))
\end{array}$$

where (see [26] for details) ∂ is the coboundary map in the long exact sequence of cohomologies associated to the short exact sequence of normal bundles

$$0 \to N_{C/X}(-1) \to N_{C/\mathbb{P}^4}(-1) \to \mathcal{O}_C(3) \to 0,$$

and as before R is the $\mathrm{H}^0 \to \mathrm{H}^1$ map of the sequence

$$0 \to \Omega_X^2 \to \Omega_{\mathbb{P}^4}^3 \otimes N_{X/\mathbb{P}^4} \to \Omega_X^3 \otimes N_{X/\mathbb{P}^4} \to 0.$$

We see that R, r_C are surjections and the cokernel of ∂ is $\mathrm{H}^1(N_{C/\mathbb{P}^4}(-1))$. Since $N_{C/\mathbb{P}^4} = \mathcal{O}_C(2)^{\oplus 3}$ (C being a complete intersection of three quadrics in $\mathbb{P}^4$), the cokernel of ∂ in the diagram above has rank 3. By Riemann-Roch, $h^0(N_{C/X}) = \deg N_{C/X} + 2(1 - g(C)) = 2 \cdot 8 - 2 \cdot 2 \cdot 2 = 8$. This implies that the map α has 3-dimensional fibres and a 5-dimensional image. But α factors via $p : \mathcal{Q} \to \mathcal{M}$ and the fibres of p are $\mathbb{P}(\mathrm{H}^0(X, E)) \cong \mathbb{P}^3$. This implies that the map $c_2 : \mathcal{M} \to \mathrm{J}^2(X)$ is quasi-finite and étale. Thus we are done. $\qquad\square$

3.2. Stable Bundles with $(c_1, c_2) = (1, 3)$.

Lemma 7. *Let E be a normalised, stable bundle of rank two on a smooth quartic threefold X with $(c_1, c_2) = (1, 3)$. Then E is ACM.*

Proof. The proof is similar to the cubic case. Let S be a very general hyperplane section as before (so S is a $K3$ surface). By the Noether-Lefschetz theorem, we have $\mathrm{Pic}(S) \cong \mathbb{Z}[\mathcal{O}_S(1)]$. Furthermore, by a result of Maruyama [19], $E_{|S}$ is stable. Hence $h^0(E_{|S}(-1)) = 0$. Recall Riemann-Roch for a quartic surface:

$$\chi(E_{|S}) = 2c_1^2 - c_2 + 4.$$

Using this we get $h^0(E_{|S}) \geq 3$.

Take any non-zero section s of E. As before, let Γ denote the intersection of S with the zero locus of s; hence Γ is a zero-dimensional subscheme of length three. From the short exact sequence

$$(6) \qquad\qquad 0 \longrightarrow \mathcal{O}_S \xrightarrow{\cdot s|_S} E_{|S} \longrightarrow I_{\Gamma/S}(1) \longrightarrow 0$$

we see that $h^0(I_{\Gamma/S}(1)) \geq 2$. From the short exact sequence

$$0 \to I_{\Gamma/S}(1) \to \mathcal{O}_S(1) \to \mathcal{O}_\Gamma(1) \to 0$$

we get a four-term exact sequence

$$0 \to \mathrm{H}^0(I_{\Gamma/S}(1)) \to \mathrm{H}^0(\mathcal{O}_S(1)) \cong \mathbb{C}^4 \to \mathrm{H}^0(\mathcal{O}_\Gamma(1)) \cong \mathbb{C}^3 \to \mathrm{H}^1(I_{\Gamma/S}(1)) \to 0.$$

Since $\mathcal{O}_S(1)$ is very ample, sections of this line bundle separate points and tangents. Hence

$$\dim \mathrm{Image}[\mathrm{H}^0(\mathcal{O}_S(1)) \to \mathrm{H}^0(\mathcal{O}_\Gamma(1))] \geq 2 \,.$$

This implies that $h^0(I_{\Gamma/S}(1)) = 2$ and hence $h^1(I_{\Gamma/S}(1)) = 1$. The long exact sequence of cohomologies associated with the exact sequence in Eq. (6) yields:

$$0 \to \mathrm{H}^1(E_{|S}) \to \mathrm{H}^1(I_{\Gamma/S}(1)) \to \mathrm{H}^2(\mathcal{O}_S) \to \mathrm{H}^2(E_{|S}) \to \cdots .$$

We have $h^2(E_{|S}) = h^0(E_{|S}^\vee) = 0$, $h^2(\mathcal{O}_S) = 1$ and from above $h^1(I_{\Gamma/S}(1)) = 1$. Hence $h^1(E_{|S}) = 0$.

The proof that $h^1(E_{|S}(1)) = 0$ is similar: using Riemann-Roch, we see that $h^0(E_{|S}(1)) \geq 11$ and so $h^0(I_{\Gamma/S}(2)) = h^0(E_{|S}(1)) - h^0(\mathcal{O}_S(1)) \geq 7$. In the four-term exact sequence

$$0 \to \mathrm{H}^0(I_{\Gamma/S}(2)) \to \mathrm{H}^0(\mathcal{O}_S(2)) \to \mathrm{H}^0(\mathcal{O}_\Gamma(2)) \to \mathrm{H}^1(I_{\Gamma/S}(2)) \to 0,$$

we have $h^0(\mathcal{O}_S(2)) = 10$ and $h^0(\mathcal{O}_\Gamma(2)) = 3$. Note that $\Gamma \subset L \cong \mathbb{P}^1$ because $h^0(I_{\Gamma/S}(1)) = 2$. This proves that the map $\mathrm{H}^0(\mathcal{O}_S(2)) \to \mathrm{H}^0(\mathcal{O}_\Gamma(2))$ is surjective. Thus $h^1(I_{\Gamma/S}(2)) = 0$ and hence $h^1(E_{|S}(1)) = 0$.

Since $h^2(E_{|S}) = 0$, this implies that the Castelnuovo-Mumford regularity of $E_{|S}$ is 2. Hence $h^1(E_{|S}(n)) = 0$ for $n \geq 1$. By Serre duality, $h^1(E_{|S}(m)) = 0$ for $m \leq -2$. Since $h^1(E_{|S}) = h^1(E_{|S}(-1) = 0$, this means that $E_{|S}$ is ACM and thus we are done as before. $\qquad\square$

For X general, one can check that the moduli space $\mathcal{M}_X(2; 1, 3)$ is smooth by using the following resolution whose proof is similar to the cubic case and hence we omit it.

Proposition 2. *Any normalised stable bundle E of rank two with $(c_1, c_2) = (1, 3)$ on a smooth quartic threefold is ACM and admits the following minimal resolution:*

$$(7) \qquad 0 \to \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}^{\oplus 3} \to E \to 0.$$

Theorem 6. *The moduli space $\mathcal{M}_X(2; 1, 3)$ is isomorphic to $F(X)$, the Fano variety of lines in X. Hence, for X general, this moduli space is a smooth irreducible curve.*

Proof. The proof of the isomorphism is similar to the proof of Theorem 3. When X is general, $\mathrm{H}^1(N_{C/X}) = \mathrm{H}^2(X, \mathcal{E}nd E) = 0$. Thus $F(X)$ is also smooth. By Riemann-Roch, it now follows that $\dim F(X) = 1$. Irreducibility is well known (see [9]). $\qquad\square$

3.3. **Bundles with $(c_1, c_2) = (0, 2)$.** By reasoning as above, it is easy to see that such bundles are ACM and admit the following minimal resolution:

$$(8) \qquad 0 \to \mathcal{O}_{\mathbb{P}^4}(-4) \oplus \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \to \mathcal{O}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \to E \to 0.$$

Theorem 7. *There is an isomorphism*

$$\mathcal{M}_X(2; 0, 2) \cong F_2(X),$$

where $F_2(X)$ is the Hilbert scheme of plane conics in X. Hence, for X general, the moduli space $\mathcal{M}_X(2; 0, 2)$ is a smooth irreducible surface.

Proof. ¿From the resolution of E, it is clear that E has unique section whose zero locus is a plane conic C. So the map

$$\mathcal{M}_X(2; 0, 2) \to F_2(X)$$

is just the one which takes $[E] \mapsto [C]$. Just as in the cubic case, here too $\mathrm{Ext}^1(I_{C/X}, \mathcal{O}_X) \cong \mathbb{C}$. Thus this map is an isomorphism. The vanishing of $\mathrm{H}^2(X, \mathcal{E}nd(E))$ follows from results of [20]. It follows then that $F_2(X)$ is smooth i.e. $h^1(N_{C/X}) = 0$. By Riemann-Roch, we have $h^0(N_{C/X}) - h^1(N_{C/X}) = \deg N + 2(1 - g)$. Since $N_{C/X} \cong E_{|C}$, we have

$$\deg N_{C/X} = c_1(E)c_2(E) = 0 .$$

Finally, the genus of C is 0. Thus we have that $F_2(X)$ is of dimension 2. Irreducibility of $F_2(X)$ has been proved in [10], smoothness follows from Lemma 6. $\qquad\square$

3.4. **Bundles with $(c_1, c_2) = (-1, 1)$.** Recall that a rank two vector bundle A with first Chern class 0 or -1, is *unstable* if and only if $h^0(A) = 0$ (see [24]). It is easy to check using Grothendieck-Riemann-Roch that normalised, indecomposable ACM bundles of rank two with $c_1 = -1$ have a unique section whose zero locus is a line i.e., $c_2 = 1$. Hence these bundles are unstable. In fact, one can also check that any such bundle is of the form $G(1)$ where for E as in section 3.2, G is defined by the exact sequence

$$0 \to G \to \mathcal{O}_X(-2) \oplus \mathcal{O}_X^{\oplus 3} \to E \to 0.$$

Geometrically speaking, if we start with a line $\ell \subset X$, then $G(1)$ is the rank two bundle which corresponds to ℓ via the Serre construction. This can be seen as follows. Let l_1, l_2, l_3 be linear forms on $\mathbb{P}^4$ whose zero locus is ℓ. Since $\ell \subset X$, the quartic polynomial f defining X is of the form

$$f = \sum_{i=1}^{3} l_i c_i,$$

where the c_i's are cubic polynomials. Let $\mathbb{P}^2 \subset \mathbb{P}^4$ be the plane defined by $l_1 = 0 = l_2$. Then $\mathbb{P}^2 \cap X$ is a reducible curve $\ell \cup C$ where C is the cubic curve given by $l_1 = l_2 = c_3 = 0$. $G(1)$ and E are the vector bundles which via Serre's construction correspond to ℓ and C respectively.

4. An Application

4.1. **Lagrangian Fibrations and complete integrable systems.** The results of this section are inspired by the observations of Beauville in [2]. Let X be a smooth cubic or quartic threefold. Let $S \in |K_X^{-1}|$ be a general member so that S is a smooth $K3$ surface. By [19], $E_{|S}$ is stable since $\dim X > \operatorname{rank} E$. Let $\mathcal{M}_X$ (respectively, $\mathcal{M}_S$) be any of the moduli spaces of stable vector bundles of rank two on X (respectively, S) discussed above and $r : \mathcal{M}_X \to \mathcal{M}_S$ be the restriction map. Mukai [23] has proved that $\mathcal{M}_S$ is smooth and carries a symplectic structure. We have an exact sequence

$$0 \to (\mathcal{E}nd E) \otimes K_X \to \mathcal{E}nd E \to \mathcal{E}nd E_{|S} \to 0.$$

Since $\mathrm{H}^1(X, \mathcal{E}nd E) \cong \mathrm{H}^2(X, \mathcal{E}nd E) = 0$, on taking cohomology we get an exact sequence

$$0 \to \mathrm{H}^1(X, \mathcal{E}nd E) \xrightarrow{r_*} \mathrm{H}^1(X, \mathcal{E}nd E_{|S}) \xrightarrow{r^*} \mathrm{H}^1(X, \mathcal{E}nd E)^\vee \to 0,$$

where r_* is the tangent map to r at $[E]$ and r^* its transpose with respect to the symplectic form. Therefore r_* is injective and its image is a maximal isotropic subspace. As a result, we have the following

Proposition 3. $\mathcal{M}_X$ *embeds as a Lagrangian submanifold of* $\mathcal{M}_S$.

As mentioned in [2], if we fix an S as above and consider the linear system of hypersurfaces containing S, we get a Lagrangian fibration (= complete integrable system) as follows: Consider the fibration $\mathcal{M}_S \to \Pi$ where Π is the linear system of hypersurfaces in $\mathbb{P}^4$ containing a given X. The fibre of this map is $\mathcal{M}_X$.

5. Conclusions

The results proved here give us a fairly complete understanding of ACM vector bundles of rank two on general smooth hypersurfaces in $\mathbb{P}^4$. As mentioned in the introduction, in the case when X is a general smooth quintic threefold in $\mathbb{P}^4$, there are only finitely many isomorphism classes of rank two indecomposable ACM bundles on X. In other words, there are only finitely many ways in which a general smooth quintic can be realised as the Pfaffian of a minimal skew-symmetric matrix of size $2k \times 2k$ for $k > 1$. It would be interesting to compute this number and understand its geometric significance. In the case when the matrix has linear entries, this number is "an instance of the holomorphic Casson invariant" (see [1]).

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